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Mass transport mechanisms in partially stratified estuaries

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A number of mechanisms which transport dissolved constituents and maintain the salt balance in estuaries are described. Although some of these have already been analysed, two which appear to be the most important in many real estuaries have received little previous attention. The Mersey Estuary is used as an example in which to estimate the amount of mass transported by each mechanism. The computations show in part how little is known and how much hypothesis is still required in spite of the number of previous estuarine studies; one may conclude, however, that in many real estuaries the most important mass transport mechanism is the net (non-tidal) transverse circulation, which is induced in part by the boundary geometry and in part by the longitudinal density gradient.

1. Introduction

Most large estuaries serve as receptacles for human and industrial waste, their transport capacity providing a convenient method of disposal to the ocean. Whatever one's ecological or aesthetic point of view, economics dictates that for the foreseeable future some wastes are bound to enter estuaries; the engineer's problem is to determine the capacity of the estuary to transport waste and the concentration of constituents which will result from a given level of discharge.

In this paper we are concerned with mass transport in partially mixed and vertically homogeneous estuaries of the coastal plain type, as defined by Pritchard (1967). Included are many of the world's most important estuaries, such as the Thames in England and the Delaware in the United States. In the past two decades coastal plain estuaries have received a great deal of study, as described, for instance, in a comprehensive review by Bowden (1967). Nevertheless, no analytical method has been found for predicting even the order of magnitude of the bulk mass transport coefficient. Indeed it has been argued that the complexities of estuarine flow defy analysis and that the only useful approach is empirical and observational. From an engineering point of view, however, one must know how to forecast the effects of such engineering works as dredging and diking, and to do this one must have some analytical understanding of what causes mass transport.

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The basic principles of dispersion in shear flow were first pointed out by Taylor (1954) in a study of dispersion in pipes. Taylor's analysis was applied to steady open channel flow by Elder (1959), who found that $D = 5{\cdot}93du^*$, where D is the coefficient in a one-dimensional mass transport equation, d is the depth of flow and u^* is the shear velocity. Elder's analysis depends on the effect of the vertical velocity profile and assumes constant flow conditions across the channel. His result does not describe natural channels, for which Fischer (1967, 1968*a*) showed that the transverse velocity profile produces much larger coefficients than those given by Elder's result. Fischer's method has given reasonable predictions of dispersion coefficients in a number of small streams (Fischer 1968*b*) and in the Missouri River in a reach where the width is about 200 m and the dispersion coefficient about 1000 times that predicted by Elder's result (Yotsukura, Fischer & Sayre 1970).

Modifications of Elder's result have been used in estuaries by, among others, Bowden (1963) and Harleman (1966). The predicted dispersion coefficients have been, perhaps not surprisingly in view of the natural stream results, consistently lower than coefficients observed in nature. Both investigators claimed that the larger coefficients observed in real estuaries are caused by a net steady vertical circulation induced by the longitudinal salinity gradient and sometimes known as the 'gravitational circulation'. Little thought seems to have been given to the possible existence of transverse circulations; indeed most laboratory and analytical studies have been limited to flows of constant depth and effectively infinite width, specifically excluding the possibility of transverse variations.

In this paper we begin by showing how the effect of net vertical circulations can be predicted from previous experimental and analytical studies. We use the Mersey Estuary, for which many data are available, as a typical example and show that the net vertical circulation does not induce a dispersion coefficient of the observed magnitude. We then investigate mechanisms which may generate transverse circulations, and show that in many estuaries transverse circulations are likely to be more important than vertical ones.

2. Modes of mass transport in estuaries

Much of the present work in estuaries has been based on averaging of currents and concentration distributions over the tidal cycle. The results of these analyses are, of course, incomplete because they fail to include mass transport effects caused by the instantaneous tidal motion. Analyses, to be discussed in detail later, have also considered in isolation the effects of the unsteady component of the velocity profile. To show the relationship between the previous analyses we begin with a rigorous decomposition of the total mass transport in an estuary into the parts caused by the tidal cycle averaged motions and the parts caused by the fluctuations from them.

2.1. *Decomposition of the velocity and concentration profiles*

The instantaneous cross-sectional velocity profile may be divided into a number of components, as was done, for instance, by Bowden (1965) and Hansen (1965).

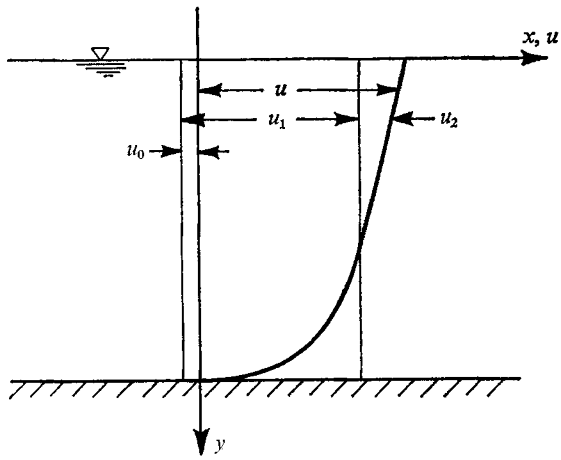

FIGURE 1. Decomposition of a two-dimensional velocity profile into three components.

Both these decompositions are included in the more general scheme to be given here. Consider first the two-dimensional profile shown in figure 1. The velocity observed at a given point can be written as the average over the cross-section at the time of observation plus the deviation therefrom. The average over the cross-section can be further divided into a tidal cycle average plus a temporal deviation. Using an overbar to denote a cross-sectional average and angle brackets to denote an average over the tidal cycle, we can write

$$u = u_0 + u_1 + u_2, \tag{1}$$

in which u_0 and u_1 are defined by $u_0 = \langle \overline{u} \rangle$ and $u_1 = \overline{u} - u_0$; by virtue of these definitions $\langle u_1 \rangle = 0$ and $\overline{u}_2 = 0$. Note that a fourth term representing the fluctuation in u due to short-term turbulent fluctuations could have been added to (1). Longitudinal turbulent fluctuations have been neglected throughout this paper, however, because their effect in producing longitudinal mass transport is known to be small compared to other mechanisms.

As shown in figure 2, the cross-sectional deviation u_2 can be further divided into its tidal cycle average u_s and a temporal deviation u'. Thus we can write

$$u_2 = u_s + u', \tag{2}$$

in which $u_s = \langle u_2 \rangle = \langle u \rangle - u_0$, and by virtue of this definition $\overline{u}_s = 0$, $\overline{u'} = 0$, and $\langle u' \rangle = 0$. The quantity $u_s + u_0$ is what is often referred to in estuaries as the 'gravitational circulation', i.e. the net steady circulation of water landward along the bottom of the channel and seaward near the surface. The velocity profiles shown in figure 1 and 2 are typical of profiles observed on a flooding tide in a partially stratified estuary, where the seaward direction is to the left.

Figures 1 and 2 are drawn in two dimensions for clarity, and the mean values over the vertical of u_2, u_s and u' are shown equal to zero as would be necessary in a purely two-dimensional flow. To discuss real estuaries it is necessary to consider the three-dimensional profile shown in figure 3 and to further subdivide both

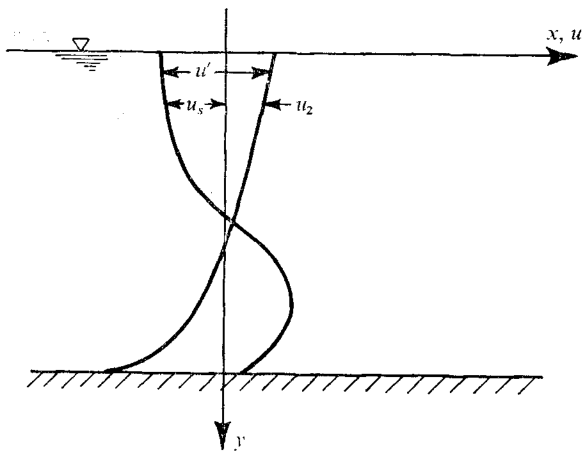

FIGURE 2. Decomposition of u_2 into a steady and a fluctuating component.

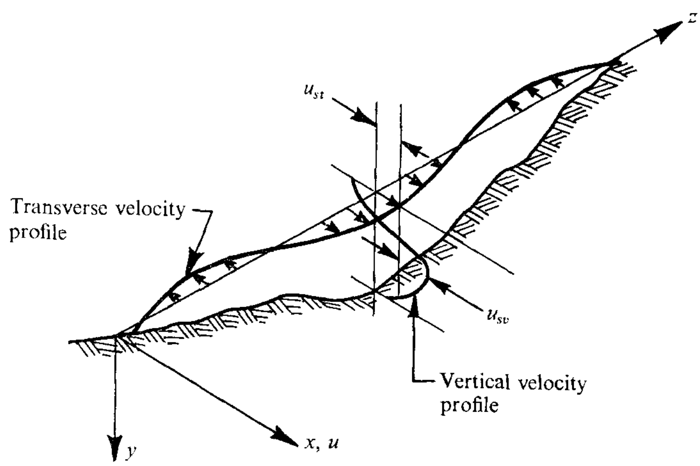

FIGURE 3. Decomposition of u_s into transverse and vertical profiles.

u_s and u' into averages taken over a given vertical and variations therefrom. Thus we can write

$$u_s = u_{st} + u_{sv}, \tag{3}$$

$$u' = u'_t + u'_v. \tag{4}$$

The subscripts t and v stand for variations in the transverse and vertical directions respectively. u_{st} and u'_t are transverse velocity profiles, defined as the means of u_s and u' respectively over the depth at a given transverse location; u_{sv} and u'_v are vertical variations from the local vertical mean. Note that in a three-dimensional flow the average of u_s or u' over any particular vertical need not be zero, but the means of u_2, u_s and u' over the entire cross-section must still be zero because of the way the terms are defined.

Combining the above equations we have as a general decomposition of the point velocity

$$u(x,y,z,t) = u_0(x) + u_1(x,t) + u_{st}(x,z) + u_{sv}(x,y,z) + u'_t(x,z,t) + u'_v(x,y,z,t). \tag{5}$$

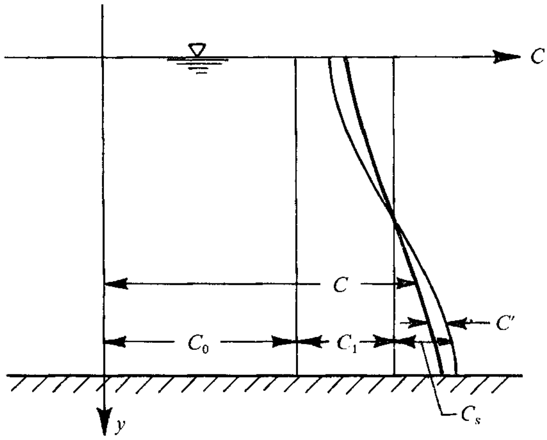

FIGURE 4. Decomposition of a two-dimensional concentration profile into four components.

In this equation x is the co-ordinate positive landward along the axis of the estuary, y is the vertical co-ordinate positive downward, z is the transverse co-ordinate and t is time. The orientation of y and z are in the sense most common in hydraulic engineering, which unfortunately is the reverse of what is common in oceanography.

The concentration profile may be written in a similar way as

$$C(x,y,z,t) = C_0(x) + C_1(x,t) + C_2(x,y,z,t), \tag{6}$$

in which

$$C_2(x,y,z,t) = C_{st}(x,z) + C_{sv}(x,y,z) + C'_t(x,z,t) + C'_v(x,y,z,t) \tag{7}$$

and, as before, $C_0 = \langle \bar{C} \rangle$, $C_1 = \bar{C} - C_0$ and $\bar{C}_2 = 0$.

Figure 4 shows a typical concentration profile in a purely two-dimensional flow; the transverse and vertical profiles for a three-dimensional flow, C_{st}, etc., are defined in exactly the same way as the analogous quantities u_{st}, etc. It should be noted, however, that the relative magnitudes of the terms in the concentration and velocity profiles are usually not the same. u_1 is usually much larger than u_0, whereas C_1 depends on the length of the tidal excursion and the longitudinal concentration gradient. If C refers to salinity in a partially stratified estuary, as illustrated in figure 4, C_1 is often much smaller than C_0. Similarly u' is usually much larger than u_s, but C' is often much smaller than C_s.

2.2. *Decomposition of the mass transport*

The tidal cycle average of the mass transport along the axis of an estuary is given by

$$\dot{M} = \frac{1}{T}\int_0^T \int_A uC\,dA\,dt, \tag{8}$$

in which T is the tidal period and A is the cross-sectional area. Hansen (1965) wrote A as the sum of a tidal cycle average A_0 and the temporal deviation from this, A_1, and obtained

$$\dot{M} = A_0 u_0 C_0 + C_0\langle A_1 u_1 \rangle + A_0 \langle u_1 C_1 \rangle + u_0 \langle A_1 C_1 \rangle + \langle A_1 (u_1 C_1)' \rangle + \langle A u_2 C_2 \rangle, \tag{9}$$

in which $(u_1C_1)'$ is the deviation of u_1C_1 from its tidal cycle mean. The first two terms may be combined to give C_0Q_f, where Q_f is the discharge of water into the estuary upstream of the cross-section.† The fourth and fifth terms are usually small, as demonstrated by Hansen for the Columbia Estuary. For the remainder of this paper we shall assume that A_1/A_0 is small. Now, considering the balance of salt in an estuary in which the salinity distribution is in equilibrium the salt balance equation is simply $\dot{M} = 0$ and (9) becomes

$$0 = C_0Q_f + A_0(\langle u_1C_1\rangle + \langle\overline{u_2C_2}\rangle). \tag{10}$$

We now assume, with justification to be given later, that a dispersion coefficient representing mass transport by the correlation of velocity and concentration gradients can be defined by

$$\langle\overline{u_2C_2}\rangle = D(dC_0/dx), \tag{11}$$

i.e. D is defined as a Fickian coefficient for diffusion of the time- and space-averaged concentration down its gradient. This is, of course, equivalent to asserting that C_0 follows a one-dimensional diffusion equation of the form $\partial C_0/\partial t + U_f\partial C_0/\partial x = D\,\partial^2C_0/\partial x^2$, if the term containing $\langle u_1C_1\rangle$ is neglected.

Using the decompositions of u_2 and C_2 and noting that because of the way the terms have been defined all cross-products of transverse and vertical variations are zero, as are all cross-products of steady terms with time-varying terms, (11) becomes

$$\begin{aligned} D &= \frac{1}{(dC_0/dx)}[\overline{u_{st}C_{st}} + \overline{u_{sv}C_{sv}} + \langle\overline{u_t'C_t'}\rangle + \langle\overline{u_v'C_v'}\rangle] \\ &= D_1 + D_2 + D_3 + D_4, \end{aligned} \tag{12}$$

where $D_1, \ldots, D_4$ are defined for reference in subsequent sections.

The four terms $D_1, \ldots, D_4$ represent respectively mass transport by the transverse net circulation, the vertical net circulation, the transverse oscillatory shear and the vertical oscillatory shear. Of these terms the components of the second were analysed by Hansen & Rattray (1965) and the third and fourth by Holley, Harleman & Fischer (1970). These analyses will be discussed in more detail in §§ 4, 5 and 6, and orders of magnitude will be estimated for each term for a real estuary. In § 6 we shall also analyse the first term, which has not been treated before.

3. Description of a prototype estuary

In the following sections we shall want to compute the magnitudes of each term in (12) for a real estuary. One can hardly speak of a typical estuary because

† Pritchard (1958) points out that $Q_f = A_0u_0 + (A_1u_1)$ and notes correctly that the second term is often much larger than the first (and of opposite sign); hence u_0 is correspondingly larger than the fresh-water discharge velocity $U_f = Q_f/A_0$. The average velocity of a fluid particle is U_f, because the average water particle must move a distance Q_fT/A_0 down the estuary during a tidal cycle to make room for the new water coming in. The larger velocity u_0 is one that could be measured at a fixed cross-section, but it is not relevant to a study of mass transport and its designation as the non-tidal drift by Pritchard and others seems a misleading choice of terms.

Maximum depth (d)	19·5 m
Width (b)	1300 m
Fresh-water discharge (Q_f)	80 m^3/s
Cross-sectional area (A)	19 000 m^2
Fresh-water discharge velocity (U_f)	0·0042 m/s
Mean tidal range	6 m
Longitudinal salinity gradient	$2{\cdot}7 \times 10^{-4}$ ‰/m
R.m.s. tidal velocity (U)	0·8 m/s
Shear velocity (u^*)	0·04 m/s
Vertical mixing coefficient for mass (ϵ_y)	0·002 m^2/s
Vertical mixing coefficient for momentum (E_y)	0·004 m^2/s
Length of approximately uniform channel	9 km
Vertical salinity variation ($\delta S/S$)	0·042
Observed longitudinal dispersion coefficient (D), two values	161 and 360 m^2/s

TABLE 1. Values of the parameters used in the computations

geometry and degree of stratification vary so widely, but the Mersey Estuary in England is typical of many and will serve as a useful example. Bowden & Gilligan (1971) have given the results of density and velocity measurements in the Egremont section of the Mersey on 13 occasions, and these data are supplemented by the results of a model and prototype study by Price & Kendrick (1963) and previous reports by Bowden (1965, 1967). Table 1 gives the values of various parameters which will be used in subsequent computations. In this table the value of the fresh-water discharge is the median given by Bowden & Gilligan (1971); the longitudinal salinity gradient is a median of the measured values as given by Bowden in personal communication; the r.m.s. tidal velocity and the value of $\delta S/S$ are medians of the values given in table 3 of Bowden & Gilligan (1971); the vertical mixing coefficient for mass is the mean of five values given by Bowden (1965); the ratio between vertical mixing coefficients for mass and momentum was suggested by Bowden in a personal communication as being a representative value for all measurements. The value of the shear velocity has been estimated by the author to be $\frac{1}{20}$ of the r.m.s. tidal velocity, on the basis of the friction coefficient used by Bowden. The two observed values of the longitudinal dispersion coefficient are given by Bowden (1967).

4. The vertical oscillatory shear

4.1. *Taylor's analysis of dispersion in shear flow*

Taylor's (1954) result for dispersion in a steady uniform shear flow may be written for a generalized cross-section as

$$D = k\overline{u}^2 l^2/\epsilon, \tag{13}$$

in which l is a characteristic length of the cross-section, ϵ is a coefficient for mixing across the section and k is a proportionality constant obtained by integration of the velocity and concentration profiles across the channel. u is the velocity relative to the cross-sectional mean, which for steady flow is u_s as defined in § 2.

Taylor's analysis for pipe flow gives $k = 0{\cdot}054$, while Elder's analysis for open channel flow gives $k = 0{\cdot}067$. Fischer (1969) tested the sensitivity of k to cross-sectional shape; for a triangular shape typical of rivers he found $k = 0{\cdot}065$, while for a typical estuary section with one deep portion and the rest shallow he found $k = 0{\cdot}16$. Apparently the order of magnitude of k is insensitive to cross-sectional shape or velocity distribution.

The analysis leading to (13) provides the basis for most of the estimates to be given in this paper. It should be remembered that the analysis is based on an asymptotic equilibrium between cross-sectional mixing and longitudinal advection which may not be reached in estuaries. Estuarine channels are generally not uniform and the time required for cross-sectional mixing is often too long to allow local formation of the equilibrium. Nevertheless, (13) has yielded accurate results in rivers which are locally non-uniform; its use seems reasonable so long as time is available for formation of an overall approximate equilibrium which, as given by Saffman (1960), requires only that $L/U_f \gg b^2/8\epsilon_z$, where L is the length of the channel, $U_f = Q_f/A$ and ϵ_z is the transverse mixing coefficient. For the Mersey $8\epsilon_z L/U_f b^2 \simeq 5$, using a value of ϵ_z to be justified in § 6. Hence it seems reasonable to expect that (13) will give results of the correct order of magnitude.

4.2. *Application of Taylor's analysis to the instantaneous tidal motion*

It was pointed out in the introduction that when Taylor's analysis is applied to a homogeneous two-dimensional shear flow with a logarithmic velocity distribution the result is that $D = 5{\cdot}93\, du^*$. Bowden (1965) tried to model the instantaneous tidal motion by assuming that the flow profile is logarithmic but oscillatory in time; he found analytically that in oscillatory flow the dispersion coefficient should be one half of that in the equivalent steady flow at the amplitude of the oscillation. For the Mersey this would give a dispersion coefficient of approximately $5\,\mathrm{m^2/s}$, a much lower value than those observed. Bowden suggested that the explanation lay in the observed value of ϵ_y, which is reduced by the stratification to about $\frac{1}{30}$ of the value one would expect in a homogeneous flow. Hence by (13) the dispersion coefficient should be increased by a factor of 30, yielding a coefficient in the observed range.

Bowden's analysis, however, failed to allow for the possibility of a phase shift between the velocity and concentration profiles. Holley *et al.* (1970) showed that the average of dispersion caused by an oscillating flow is less than that for an equivalent steady flow by a factor depending on the ratio $T' = T/T_c$, where T is the period of oscillation and $T_c = l^2/\epsilon$ is the time scale for cross-sectional mixing. For T' greater than unity the reduction is nil, but for T' less than $0{\cdot}2$ the results of Holley *et al.* may be approximated by

$$D/D_0 = 3T'^2, \tag{14}$$

where D_0 is the dispersion coefficient for the equivalent steady flow. For very small T' the concentration and velocity profiles have a phase shift approaching 90°, so the covariance averaged over the tidal cycle gives zero dispersion. In estuaries with no vertical density gradient a typical value of T_c for vertical mixing is on the order of 20 min, so T' is very large. In partially stratified estuaries, on the other hand, vertical mixing is hindered; T' is often much less than unity and the result of the analysis of Holley *et al.* must be used to estimate the effect on longitudinal dispersion.

For the Mersey, using the value for ϵ_y measured by Bowden of 0·002 m²/s, the value of T' for vertical mixing is approximately 0·24. Hence Taylor's analysis, as applied only to the instantaneous vertical velocity profile and modified to account for the smaller vertical mixing coefficient and the effect of oscillation, gives a value for D_4 as defined in (12) of approximately 23 m²/s. This is the result of only the vertical profile of the instantaneous tidal motion.

5. The net vertical circulation

A longitudinal density gradient will drive a net vertical circulation because of the longitudinal gradient of hydrostatic pressure. Most investigators have claimed that this circulation is the primary mechanism for maintenance of the salt balance in partially stratified estuaries. Widely quoted evidence for the importance of the vertical circulation is given by Pritchard (1954) for the James River, Virginia. Pritchard contrasted the cross-sectional variation of two terms which in our notation are $(u_0+u_s)(C_0+C_s)$ and $\langle u'C'\rangle$, and concluded that because the first of these varies much more over the cross-section than the second it must represent the more important mechanism for mass transport. The contrast may be misleading, however, because the first term includes at each depth the value of u_sC_0, which is locally much bigger than $\langle u'C'\rangle$ but whose average over the cross-section is zero. If one subtracts from each of Pritchard's tabulated results the value of u_sC_0 his conclusion is less apparent. In this section we shall conclude, in contrast to Pritchard's result, that in many estuaries the net vertical circulation is not a sufficient mechanism to maintain the salt balance.

5.1. *Analytical studies*

Analytical studies of the vertical circulation have been given by Hansen & Rattray (1965, 1966) and Prych (1970). Hansen & Rattray found a similarity solution for the vertical net velocity and salinity profiles based on two dimensionless parameters which were empirically related to bulk parameters of the estuary. Prych obtained the same velocity profile and from it used an analysis similar to Elder's to obtain

$$D_2 = 10^{-2}(d^2/\epsilon_y)(1{\cdot}34K^2+3{\cdot}02KU_f+1{\cdot}90U_f^2), \tag{15}$$

where $K = \frac{1}{32}d^3N^2/E_y$ is the order of magnitude of the induced velocity, E_y is the vertical mixing coefficient for momentum and $N^2 = -(g/\rho)\,d\rho/dx$. For the Mersey Narrows N^2 is approximately $1{\cdot}9\times10^{-6}\,\mathrm{s}^{-2}$, giving a value of K of 0·11 m/s. Bowden & Gilligan (1971) report net surface velocities in the Mersey of between 0·1 and 0·2 m/s, in reasonable agreement with the analysis. A surface velocity of approximately 0·1 m/s can also be obtained from Hansen & Rattray's (1966) semi-empirical plot of surface to mean velocity *vs.* a Froude number. Using $K = 0{\cdot}11$ m/s yields a dispersion coefficient $D_2 = 32\,\mathrm{m}^2/\mathrm{s}$ as the result of the tidal-cycle-averaged velocity profile. The total result of vertical variations of velocity is the sum of this result and the one already obtained for the effect of the instantaneous velocity profile, or approximately $D = 55\,\mathrm{m^2/s}$.

5.2. *Dimensional analysis*

Prych's solution is not explicit in the sense that the values of N^2, E_y and ϵ_y depend on the state of stratification and therefore on D. An alternative approach is to carry out experiments and correlate the results against the proper dimensionless numbers. If we consider an infinitely wide channel of constant depth the dispersion coefficient can be assumed to depend only on d, u^*, U, Q_f/b and $g\Delta\rho/\rho$, where U is the r.m.s. tidal velocity, Q_f/b is the discharge of fresh water per unit width and $\Delta\rho$ is the density difference between fresh and ocean water.

Ellison & Turner (1960), in a study of mixing across an interface in an inclined channel, showed that the physically important parameters are the velocity and the input of buoyancy per unit width. For estuaries the buoyancy input per unit width is $(\Delta\rho/\rho)\,gQ_f/b$, and the equivalent of what Ellison & Turner called the pipe Richardson number is what we shall call the estuarine Richardson number, $Ri_E = g\Delta\rho Q_f/\rho b U^3$. For a second dimensionless number it is convenient to choose a Froude number based on the fresh-water discharge velocity like that used by Hansen & Rattray, $F = Q_f/bd(gd\Delta\rho/\rho)^{\frac{1}{2}}$. We can now write

$$D/du^* = f(Ri_E, F, U/u^*). \tag{16}$$

It may be of interest to digress for a moment and notice the physical meaning of Ri_E and F. Hansen & Rattray (1966) give a plot of $\delta S/S$ *vs.* u_{surf}/U_f, where δS is the salinity difference between surface and bottom and u_{surf} is the net surface velocity, in terms of F and a mixing parameter $P = Q_f/bdU$. The plot is semi-empirical, being derived from a combination of their analysis with admittedly sketchy data from six real estuaries. They show that u_{surf}/U_f depends solely on F but that $\delta S/S$ depends on both F and P. If their result is replotted in terms of Ri_E and F, as in figure 5, $\delta S/S$ is found to depend primarily on Ri_E and only slightly on F. Figure 5 also includes some laboratory and field results not considered by Hansen & Rattray, all of which substantiate the previous empirical correlation. It appears that the degree of stratification in an estuary depends primarily on Ri_E, while the magnitude of the vertical circulation depends primarily on F.

5.3. *Experimental results*

Ippen & Harleman (1961) have described the results of a careful study of mass transport in oscillating flow in a rectangular laboratory channel 327 ft long and 0·75 ft wide at the Waterways Experiment Station in Vicksburg, Mississippi. The experiments have been summarized by Ippen (1966) and re-analysed by Harleman & Ippen (1967), and will be referred to henceforth as the WES studies (including all these reports). Because of the geometry of the channel transverse variations appear to have been unimportant (as was intended); hence the experiments can be used to determine the isolated effect of vertical currents. During the course of the experiments the fresh-water discharge, ocean density and tidal

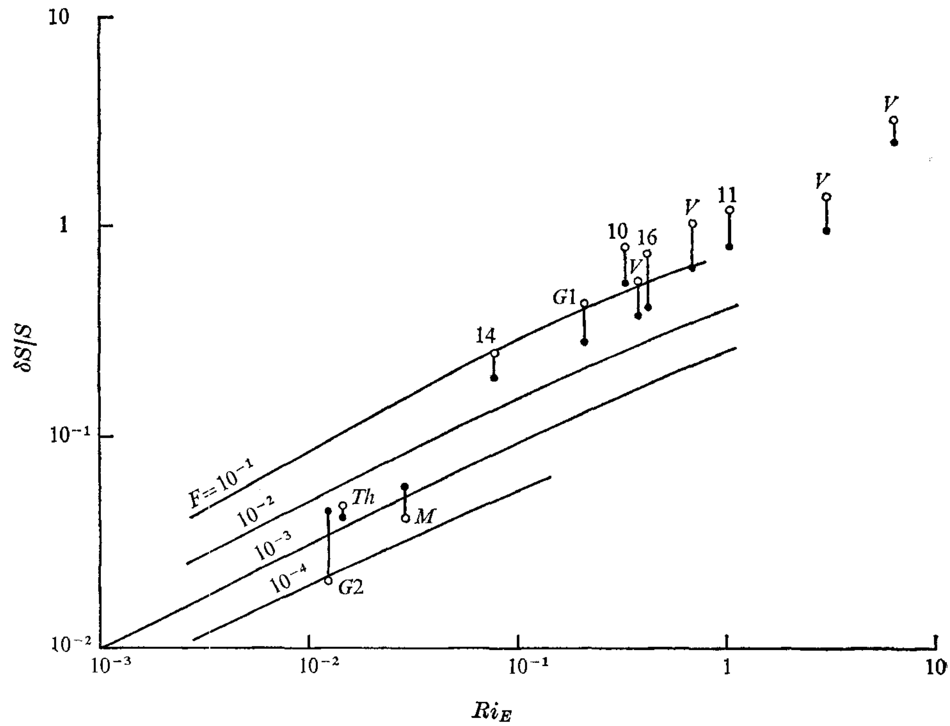

FIGURE 5. Relative stratification *vs.* estuarine Richardson number and Hansen & Rattray's Froude number. Lines of constant Froude number are as given by Hansen & Rattray (1966). ♀, $\delta S/S$ *vs.* Ri_E as given by observed data; ♦, value of $\delta S/S$ corresponding to observed values of Ri_E and F according to the results of Hansen & Rattray. Numbers above points are run numbers of the WES experiments. Other points are identified as follows: *G*1, Gironde Estuary at point PK89 on 13 May 1969; *G*2, Gironde Estuary at point PK78 on 17 December 1968, both values from data given by Bonnefille (1970); *Th*, Thames Estuary, from data given by Inglis & Allen (1957); *M*, Mersey Estuary; *V*, Vellar Estuary from data given by Dyer & Ramamoorthy (1969).

velocities were varied independently to yield a reasonable range of Richardson and Froude numbers; the depth, however, was held constant at 0·5 ft and one may assume that the friction factor was approximately constant. Thus the experiments can be used to determine the dependence of D/du^* on Ri_E and F, but not on U/u^*.

The WES results are replotted in figure 6. In this figure D_0 is the dispersion coefficient in a constant-density estuary, obtained by putting dye but no salt in the ocean, and D is the coefficient observed for corresponding tidal conditions and a saline ocean. Thus D_0 results from the oscillatory shear terms only, whereas D includes the net vertical circulation and also reflects the reduction in vertical mixing due to stratification. The investigators reported that D_0 had the value $2{\cdot}8G^{\frac{1}{3}}$, where G is the rate of energy dissipation per unit mass and the units are feet and seconds; the factor 2·8 was the same for three experiments with different tidal ranges and velocities. The result can be made properly dimensionless and compared with the analytical results of Taylor and Elder by introducing the

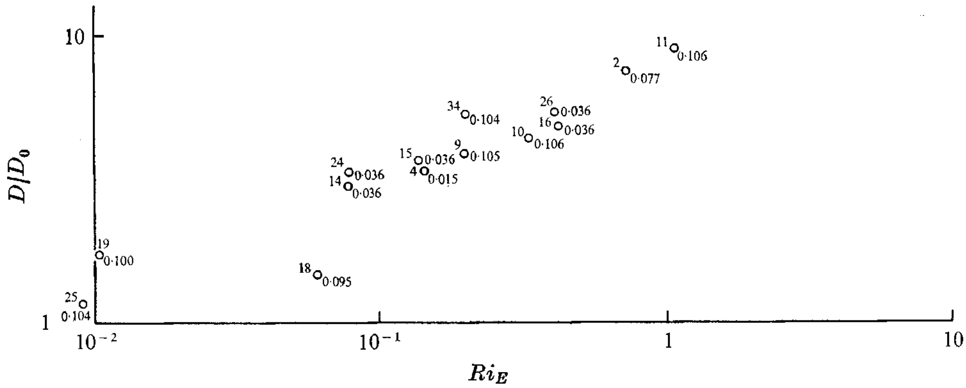

FIGURE 6. Increase in the longitudinal dispersion coefficient due to the net vertical circulation, as given by the WES studies. Numbers above left of points are run numbers; numbers below right are values of F.

depth, which was held constant, and using the relationship, which is exact for steady uniform flow, that $G^{\frac{1}{2}}d^{\frac{3}{2}} = 1{\cdot}4f^{-\frac{1}{2}}du^*$, where f is the Darcy–Weisbach friction factor. By using an assumed value of $f = 0{\cdot}05$ the experimental result is found to be equivalent to $D_0 \cong 16du^*$. The result is somewhat higher than would be predicted from Elder's theory ($5{\cdot}93du^*$), but is approximately equal to the median of 197 measurements by Fischer (1967) of dispersion coefficients in steady flows in rectangular channels. Thus in the absence of density effects the WES experiments yielded dispersion coefficients which were similar to those which have been measured in similar steady flows, and which were of the magnitude predicted by application of Taylor's theory, considering only the vertical velocity profile.

When a salinity gradient has been introduced figure 6 shows that, within reasonable limits, the increase in D due to density effects depends solely on Ri_E. Values of F are shown beside each point, but no systematic effect can be seen.

The results shown in figure 6 might have been anticipated by Ellison & Turner's argument, which suggests that D depends only on $Q_f g\Delta\rho/\rho b$ and not on Q_f/b and $g\Delta\rho/\rho$ separately. If that were true F would have to be deleted from (16) and dimensional analysis would assert that the increase of D/du^* due to gravitational effects does not depend on d. Such appears to be the case, but considering that (15) contains the eighth power of depth one must admit that the result is remarkable. We can only wish in retrospect that during the WES experiments the depth had occasionally been varied so that the depth dependence could be seen independently of the other parameters.

Returning to the Mersey Estuary, the computed value of Ri_E is $3{\cdot}0 \times 10^{-2}$ and figure 6 gives a value of D/D_0 of $2{\cdot}2$, giving $D = 29\,\mathrm{m^2/s}$. This is in reasonable agreement with the separate results for the vertical oscillatory shear and the vertical net circulation, although less than their sum. All three values of D are substantially below the observed range.

5.4. *Correlation with previous analyses*

The WES studies were previously correlated with different dimensionless numbers and it may be of interest to show why Ri_E is an equivalent but better parameter to describe the same results. The earlier studies used a 'stratification parameter' G/J, where $J = \Delta\rho g d U_f/\rho L$ and L is the length of the tank. The stratification parameter can be expressed in terms of the Richardson number by noting that for steady uniform flow $G = gUS$, where S is the channel slope, which by the Chezy equation is proportional to U^2/gd. Making these substitutions gives

$$G/J \propto Ri_E^{-1}(L/d). \tag{17}$$

In the experiments L/d was held constant; had it been varied it seems clear that the studies would not have yielded a unique relationship between G/J and D/D_0.

In the later analysis Harleman & Ippen (1967) define an 'estuary number' as $E = P_T U_0^2/gdU_f T$, where P_T is the tidal prism and U_0 is the maximum flood-tide velocity. This number can also be expressed in terms of Ri_E by noting that to a good approximation $P_T \propto U_0 bdT$. Assuming a proportionality between U and U_0 gives

$$E \propto Ri_E^{-1}(\Delta\rho/\rho). \tag{18}$$

This explains why Harleman & Ippen were able to correlate G/J with E only for those experiments having the same value of $\Delta\rho/\rho$ (see their figure 11, which gives a linear relation between E and G/J but only includes points for which $\Delta\rho/\rho \cong 0{\cdot}025$; the remaining points, which would be off the line, are not shown).

In summary, the correlations obtained with G/J and the 'estuary number' depended on certain parameters being held constant during the experiments. The estuarine Richardson number appears to express correctly the dependence of D/D_0 on all the parameters, and is therefore preferable to the dimensionless numbers used previously.

6. Transverse circulations

6.1. *The transverse oscillatory shear*

The effect of the transverse oscillatory shear can only be estimated because the transverse velocity profiles have not been measured (making a detailed measurement of a transverse profile in a real estuary is extremely difficult). To make the estimate we return to (13), in which k can be taken equal to $0{\cdot}1$, being a typical value for estuaries (Fischer 1969). The mean velocity deviation for a typical velocity profile is approximately $\frac{1}{5}$ of the mean velocity, so a reasonable estimate of $\overline{u'^2}$ is $0{\cdot}02\,\mathrm{m^2/s^2}$. l, which should be taken as the distance from the thread of maximum velocity to the most distant bank, we take to be 1000 m.

The least well understood parameter is the transverse mixing coefficient ϵ_z. In straight uniform channels of constant depth and large width-to-depth ratio experiments have established reasonably well (Prych 1970) that $\epsilon_z \cong 0{\cdot}15du^*$. Implicit in this formulation is that the scale of the horizontal eddies is determined by the depth. In estuaries larger scale eddies will be expected because of channel

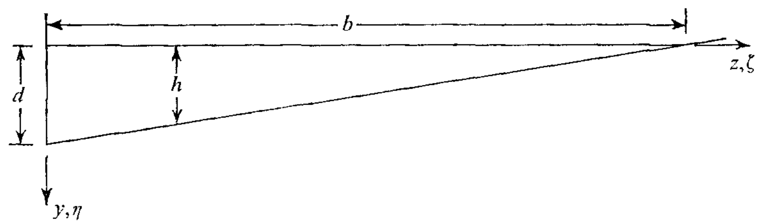

FIGURE 7. Definition sketch for the cross-section used in the analysis.

bends and interaction of the tidal currents with the irregular boundary geometry, and experimental results are often conflicting. In South San Francisco Bay, Ward & Fischer (1971) computed a value of ϵ_z between 0·25 and 0·40 m^2/s; in the Mersey, with its greater depth but otherwise similar geometry, a value of 0·50 m^2/s seems reasonable. With this value of ϵ_z and the values of l, $\overline{u'^2}$ and k already mentioned (13) and (14) yield a value of $D_3 = 6{\cdot}0\,\mathrm{m^2/s}$. The relatively small value is a reflexion of the reasonably large width of the cross-section and the correspondingly long time scale for transverse mixing.

6.2. *The transverse gravitational circulation*

Previous studies of the current induced by a longitudinal density gradient have been limited to channels of constant depth; the investigators seem to have overlooked the fact that in a cross-section of variable depth a net transverse circulation will also be generated. This is because the longitudinal pressure gradient depends on the depth; on the average the inflow will be concentrated in the deeper parts of the cross-section while the return flow is in the shallower parts.

To quantify the transverse circulation consider the simple triangular cross-section shown in figure 7. Neglecting accelerations and transverse shear stress

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(E_y \frac{\partial u}{\partial y}\right).$$

Integrating from 0 to y yields

$$\rho g y S_w + \tfrac{1}{2} g y^2 \frac{d\rho}{dx} = \rho E_y \frac{\partial u}{\partial y}, \tag{19}$$

in which S_w is the slope of the water surface, x is positive landward and $\partial\rho/\partial x$ is assumed to be a function of x only. Averaging over the tidal cycle and neglecting variations in y during the cycle gives

$$E_y \frac{\partial u_s}{\partial y} = g y \langle S_w\rangle - \tfrac{1}{2} y^2 N^2, \tag{20}$$

where $N^2 = -(g/\rho)\,d\rho/dx$ and the boundary condition is $u_s = 0$ when $y = h$. Setting $\eta = y/h$ and integrating gives

$$u_s = \frac{g}{E_y}\left[\frac{\langle S_w\rangle}{2} h^2(\eta^2 - 1) - \frac{N^2 h^3}{6g}(\eta^3 - 1)\right]. \tag{21}$$

The variation of u_s with z is implicit in the presence of h, which is a function of z. Equation (21) was also obtained by Prych (1970), who justified the assumptions used to obtain (19) more rigorously, and is also equivalent to the result given by Hansen & Rattray (1965).

The mean water slope $\langle S_w \rangle$ is determined by noting that the average, over the cross-section, of u_s must be equal to $-U_f$. The result will depend somewhat on the assumed variation of E_y; a simple reasonable assumption is that both E_y and ϵ_y depend linearly on local depth, i.e. $E_y = E_0\zeta$ and $\epsilon_y = \epsilon_0\zeta$, where $\zeta = h/d = z/b$. Setting $K_0 = \frac{1}{32}d^3N^2/E_0$ gives

$$\langle S_w \rangle = \frac{9E_0}{gd^2}(\tfrac{1}{2}U_f + K_0).$$

Substituting this into (21) and averaging over the depth yields

$$u_{st} = -\tfrac{3}{2}U_f\zeta + K_0(4\zeta^2 - 3\zeta). \tag{22}$$

The first term represents a velocity profile for the net seaward velocity and the second term is the transverse gravitational circulation. By considering only the second term (i.e. assuming $U_f \ll K$) a dispersion coefficient can be found from the formulation based on Taylor's analysis:

$$D = \int_0^b ud \int_0^{z_1} \frac{1}{\epsilon d} \int_0^{z_2} ud\, dz_3\, dz_2\, dz_1, \tag{23}$$

with the result that

$$D_1 = 0{\cdot}019K_0^2b^2/\epsilon_z. \tag{24}$$

For the Mersey this gives $D_1 = 430\,\mathrm{m^2/s}$, a significantly larger value than that computed by other mechanisms.

6.3. *Boundary-induced circulation*

A number of other mechanisms which are difficult to quantify may produce transverse circulations, for instance wind stress and the Coriolis acceleration. In addition it is quite possible that a net transverse circulation can be induced by the interaction of the tidal currents with the boundary geometry. The simplest example would be flow through a small entrance into a circular basin; the flow enters as a jet and leaves as a potential sink flow, implying a net circulation inward along the diameter and outward along the edges. In more realistic situations one often notices that flood currents are stronger in one part of an estuary and ebb currents stronger in another. Analytical solutions are unlikely to describe the current distribution in real estuarine geometries completely, but reference can be made to numerical and physical models. For instance, Price & Kendrick, in their model study of the Mersey, measured net drifts per tidal cycle which imply a transverse velocity profile with a mean velocity deviation of about 0·1 ft/s. This corresponds to a dispersion coefficient of approximately $130\,\mathrm{m^2/s}$, which represents the result of both the boundary-induced and the gravitational net transverse circulations in the model.

7. Summary and conclusions

In this paper we have tried to isolate and determine the magnitudes of the mechanisms responsible for causing the observed rates of longitudinal mass transport in partially stratified estuaries. The principle involved in Taylor's analysis of longitudinal dispersion in pipes has been used, with the hope that although the equilibrium required by that analysis may not be reached in real estuaries the results will be correct at least in order of magnitude. The Mersey Estuary has been used as an example of the magnitude of the physical parameters typical of many real estuaries, with the following results.

(i) The vertical oscillatory shear $D_4 = 23\,\mathrm{m^2/s}$.

(ii) The net vertical circulation $D_2 = 32\,\mathrm{m^2/s}$.

(iii) The effect of all vertical gradients, as shown by the WES studies, $D_2 + D_4 = 29\,\mathrm{m^2/s}$.

(iv) The transverse oscillatory shear $D_3 = 6\,\mathrm{m^2/s}$.

(v) The transverse gravitational circulation $D_1 = 430\,\mathrm{m^2/s}$.

The dispersion coefficient generated by the sum of all the mechanisms is somewhat greater than the observed range of values of 161–360 $\mathrm{m^2/s}$, suggesting that the analysis may have overestimated the effect of the transverse gravitational circulation. If the transverse gravitational circulation is neglected, however, the sum of the other mechanisms is substantially less than the observed range of coefficients.

The relative magnitudes shown for the various mechanisms are for one estuary only and are all subject to a substantial degree of hypothesis. It seems clear, however, that contrary to frequent statements in previous literature the vertical gravitational circulation is not necessarily the most important mechanism. We have shown that the ratio of the effect of the transverse to vertical gravitational circulations is of order $b^2\epsilon_y/d^2\epsilon_z$; in many real estuaries this ratio is large, and it appears that the dispersion coefficient caused by gravitational effects will be proportional to the square of the width.

Every estuary has its individual characteristics and the complexities of estuarine geometry are such that no analytical theory is ever likely to describe mass transport in estuaries completely. Instead recourse is made to numerical simulation programmes, which become ever more complex with the increasing size and speed of electronic computers. Even here, however, a complete description of the mechanics of estuarine mass transport has not yet been achieved, nor does it seem likely to be achieved in the near future. Indeed, numerical programmes which have been published up to the present time have not even considered the transverse gravitational circulation and therefore may have omitted the most important part of estuarine circulation. One conclusion of this study is that numerical programmes which include the effect of the transverse gravitational circulation should be written, and that those programmes presently in existence which do not include the transverse circulation are unlikely to describe accurately mass transport in partially stratified estuaries.

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